Average Rate of Change

-The average rate of change of a quantity over a period of time is the amount of change divided by the time it takes.

Example

Find the average rate of change of $f(x) = x^3 - x$ over $[1,3]$.

$$f(1) = 0$$
$$f(3) = 24$$

$$\frac{f(3) - f(1)}{3 - 1} = \frac{24 - 0}{2} = 12$$

Example

From a study on the population of fruit flies in a jar two sample points are taken $P(23,150)$ and $Q(45,340)$. Use these points to find the average rate of change and the slope of the secant line PQ.

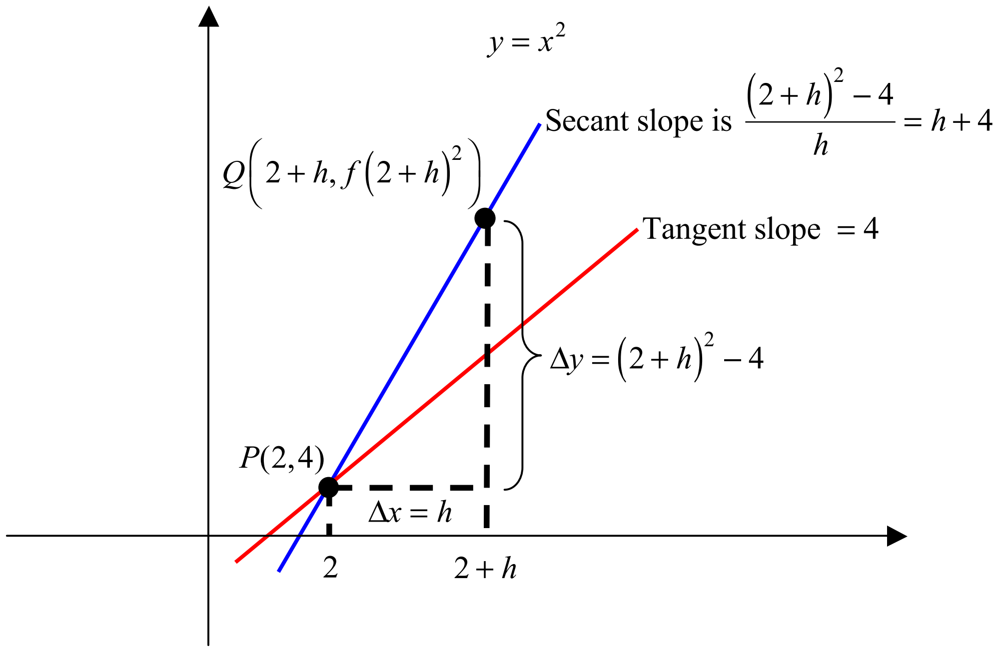
-There are 150 flies on day 23 and 340 on day 45.

Average rate of change: $\frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6$ flies/day

-The average rate of change is also the slope of the secant line from P to Q.

Tangent to a Curve

Find the slope of the parabola $y = x^2$ at the point $P(2,4)$. Write an equation for the tangent line.



-We begin with a secant line through $P(2,4)$ and a nearby point $Q\left(2+h,(2+h)^2\right)$.

-We then write an expression for the slope of the secant line and find the limiting value of this slope as Q approaches P.

$$\text{Secant Slope: } \frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - 4}{h}$$

$$= \frac{h^2 + 4h + 4 - 4}{h}$$

$$= \frac{h^2 + 4h}{h} = h + 4$$

$$\lim_{Q \to P}(\text{secant slope}) = \lim_{h \to 0}(h+4) = 4$$

-The slope of the parabola at P is 4.

-The equation of the line tangent to this point.

$$y - 4 = 4(x - 2)$$
$$y - 4 = 4x - 8$$
$$y = 4x - 4$$

Slope of a Curve at a Point

The slope of a curve $y = f(x)$ at the point $P(a, f(a))$ is the number

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided that the limit exists.

Example

Let $f(x) = \frac{1}{x}$

a) Find the slope of the curve at $x = a$.

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \bullet \frac{a - (a+h)}{a(a+h)}$$

$$= \lim_{h \to 0} \frac{-h}{ha(a+h)}$$

$$= \lim_{h \to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}$$

b) The slope will be $-1/4$ if

$$\frac{-1}{a^2} = \frac{-1}{4}$$

$$a^2 = 4$$

$$a = \pm 2$$

Normal to a Curve

Find the equation normal to the curve if $f(x) = 4 - x^2$ at $x = 1$

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{4 - (1+h)^2 - 3}{h}$$

$$= \lim_{h \to 0} \frac{4 - 1 - 2h - h^2 - 3}{h}$$

$$= \lim_{h \to 0} \frac{4 - 1 - 2h - h^2 - 3}{h}$$

$$= \lim_{h \to 0} \frac{-h(2+h)}{h} = -2$$

$$y - 3 = \frac{1}{2}(x-1)$$

$$y = \frac{1}{2}x - \frac{1}{2} + 3$$

$$y = \frac{1}{2}x + \frac{5}{2}$$

Instantaneous Rate of Change (or speed)

-An objects instantaneous speed at any time t is the instantaneous rate of change of position with respect to time t or,

$$\lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

Example

The position function of a rock $f(t) = 16t^2$. The average speed of the rock over the interval between $t = 1$ and $t = 1 + h$ seconds was

$$\frac{f(1+h) - f(1)}{h} = \frac{16(1+h)^2 - 16(1)^2}{h} = \frac{16(h^2 + 2h)}{h} = 16(h+2)$$

at $t = 1$

$$\lim_{h \to 0} 16(h+2) = 32 \ {}^{\text{ft}}\!/_{\text{sec}}$$